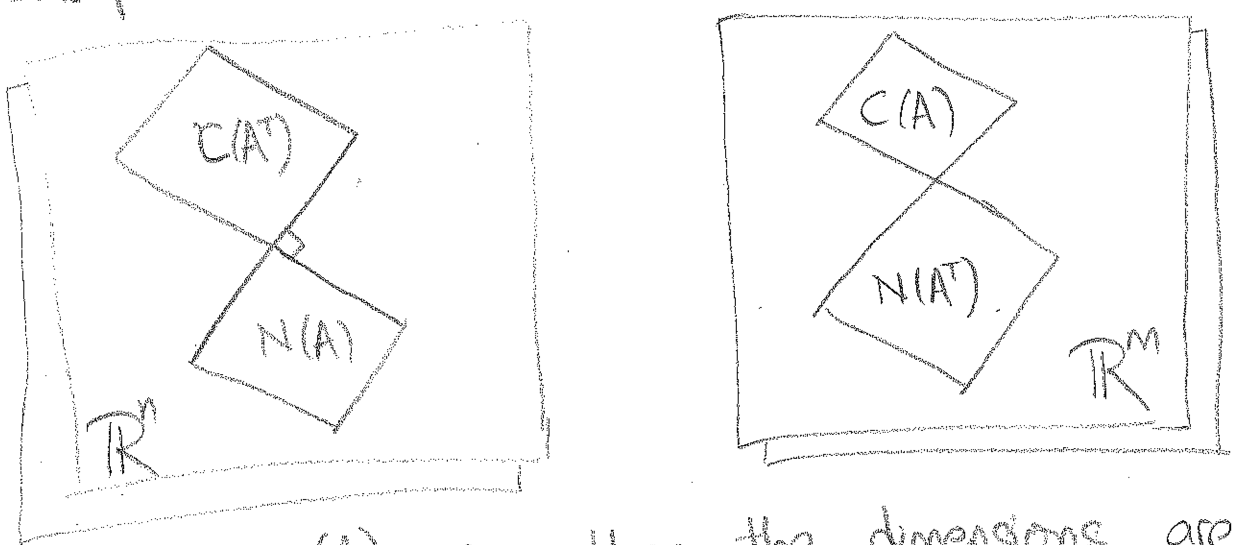
MATH 312
Lecture 22

SINGULAR VALUE DECOMPOSITION

OR, The Day we used EVERYTHING we know.

Here are two stories:

I Any $m \times n$ matrix A has four fundamental subspaces:



When $\text{rank}(A) = r$, then the dimensions are given by $\dim C(A) = r = \dim C(A^T)$ and $\dim N(A) = n - r$, $\dim N(A^T) = m - r$.

Moreover, we have ORTHOGONALITY! That is,

$$N(A)^\perp = C(A^T) \quad \text{in } \mathbb{R}^n,$$
$$C(A)^\perp = N(A^T) \quad \text{in } \mathbb{R}^m.$$

This was the first "third" of our course!

II When A is an $n \times n$ (i.e., square) matrix, then:

- its EIGENVALUES $\lambda_1, \ldots, \lambda_n$ are solutions to

$$\det(A - \lambda I) = 0$$

(← non-linear)
May be repeated / Complex!

- its EIGENVECTORS $v_1, \dots, v_n$ are NONZERO solutions to

$$Av_j = \lambda_j v_j$$ (← linear)

or, v_j in $N(A - \lambda_j I)$

- A is "diagonalizable" when $v_1, \dots, v_n$ are linearly independent. In this case, we get

$$A = SDS^{-1}, \text{ where}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

- When A is SYMMETRIC and Real, then we have nicer properties, i.e.,
  - A is diagonalizable,
  - Its eigenvalues are Real,
  - Its eigenvectors can be chosen orthonormal.

So, $\boxed{A = QDQ^T}$. This is the SPECTRAL THEOREM.

This was the second "third" of our course

The SVD gives stories (I) and (II) a common happy ending. (No, not <u>that</u> kind)

Thm

Very, very important

[SVD] Let A be any $m \times n$ matrix. Then,

$\boxed{A = UDV^T}$ i.e. (orthogonal) (diagonal) (orthogonal)

U is $m \times m$ and contains eigenvectors of AA^T

V is $n \times n$ and contains eigenvalues of A^TA

D is diagonal, with square roots of nonzero eigenvalues of AA^T AND A^TA.

There's a lot going on here, so let's chop this into pieces (like we did in Lecture 21 with the SPECTRAL THEOREM).

PART I: Eigenvalues of A^TA and AA^T are ≥ 0

When A is $m \times n$, we know that

- A^TA is $n \times n$
- AA^T is $m \times m$

and BOTH are symmetric! For example

$$(A^TA)^T = A^T(A^T)^T = A^TA.$$

A^TA transposed equals A^TA

So, the SPECTRAL THEOREM applies to both!

There's ONE MORE PIECE:

Def A matrix P is called POSITIVE SEMIDEFINITE if it is symmetric, real, and satisfies

$$x^TPx \geq 0 \quad \text{for all vectors } x.$$

eg

- Identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $x^TIx = \|x\|^2$
- Symmetric matrices w/ positive entries
- And more! The next proposition tells us:

Prop P is positive semidefinite if and only if all its eigenvalues are positive!

Pf $\Rightarrow$ (easy) If P is pos semidef and λ is an eigenvalue then $Pv = \lambda v$ for some $v \neq 0$ eigenvector.

So $v^TPv = v^T\lambda v = \lambda \|v\|^2$

So: $v^TPv \geq 0$ means $\lambda\|v\|^2 \geq 0$ so $\lambda \geq 0$ ✓.

$\Leftarrow$ (hard) Since P is symmetric, it is diagonalizable (spectral Thm!) and also, its eigenvectors $v_1, \dots, v_n$ can be chosen ORTHONORMAL. So,

$$v_i^T v_j = \begin{Bmatrix} 1, & i=j \\ 0, & i \neq j \end{Bmatrix} \longrightarrow (*).$$

These form a basis, so any x is a linear combination:

$$x = c_1 v_1 + \dots + c_n v_n.$$

Now, $\underline{x^T P x} = (c_1 v_1^T + \dots + c_n v_n^T) \, P \, (c_1 v_1 + \dots + c_n v_n)$

This creates n^2 terms of the form

$$(c_i v_i^T) \, P \, (c_j v_j)$$

$$= c_i c_j \, v_i^T P v_j$$

$$\left(\underline{\text{BUT}} \quad v_i^T P v_j = v_i^T \lambda_j v_j \right)$$

where λ_j is the eigenvalue of P corresponding to the eigenvector v_j.

So, these terms become

$$c_i c_j \lambda_j \, v_i^T v_j$$

Now use $(*)$: all but n of the terms are zero, we only get nonzero terms when $i=j$.

$$x^T P x = c_1^2 \lambda_1 + \dots + c_n^2 \lambda_n.$$

Since all the λ's are ≥ 0, so is $x^T P x$.

Why do we care? Because AA^T and $A^T A$ are POSITIVE SEMI DEFINITE, hence have ≥ 0 eigenvalues! Note,

$$x^T (A A^T) x = (A^T x)^T (A^T x) = \| A^T x \|^2 \geq 0$$

PART II | Eigenvectors of AA^T and A^TA

Let's start with A^TA: if A^TA has an eigenvalue $\lambda \geq 0$ then there must be an eigenvector $v \neq 0$, so

$$A^TAv = \lambda v \qquad (*)$$

Multiply by A on both sides:

$$A(A^TAv) = A(\lambda v)$$

$$\text{So } (AA^T)\,\underline{\underline{Av}} = \lambda\,\underline{\underline{Av}}$$

There are only <u>two</u> choices:

- Either $Av \neq 0$, so λ is an eigenvalue of AA^T also with eigenvector Av in $C(A)$,
- Or $Av = 0$ so that A^TAv is also zero, in which case $\lambda = 0$, by $(*)$ above.

What this <u>MEANS</u>, is that the

- NONZERO EIGENVALUES OF A^TA and AA^T COINCIDE!
- Eigenvectors of AA^T corresponding to nonzero eigenvalues lie in $C(A)$, and those of A^TA lie in $C(A^T)$.
- Eigenvectors of A^TA corresponding to zero eigenvalues lie in $N(A)$ and those of AA^T lie in $N(A^T)$.

Example 1

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$ What is the SVD?

STEP 1: Compute A^TA and AA^T:

$$A^TA = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Sanity check! These should both be SYMMETRIC!

STEP 2: Compute eigenvalues & eigenvectors of both.

unit (!)

A^TA has:

$\lambda_1 = 3$, $\vec{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$, $\vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\lambda_3 = 0$, $\vec{v_3} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

These should be orthonormal.

AA^T has:

$\lambda_1 = 3$, $\vec{u_1}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1$, $\vec{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

These should also be orthonormal.

These nonzero λ's should be the same.

STEP 3 Put it together!

$$D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(same shape as A)!

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$V^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

All that stuff

Finally, $$A = UDV^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \; \end{bmatrix}$$

RELATING SVD TO THE FUNDAMENTAL SUBSPACES

If A is a $m \times n$ matrix of rank r, then both A^TA and AA^T also have rank r (even though A^TA is $n \times n$ and AA^T is $m \times m$)

If the SVD of A is

$$A = UDV^T \quad \text{where}$$

$$D = \left[\begin{array}{cc|c} \lambda_1 & & 0 \\ 0 & \lambda_r & 0_{r \times (n-r)} \\ \hline \multicolumn{2}{c|}{0_{(m-r)\times r}} & 0_{(m-r)\times(n-r)} \end{array}\right]$$

Also $m \times n$ like A, but with

$(\lambda_1^2 \geq \lambda_2^2 \geq \dots \geq \lambda_r^2$ are nonzero common eigenvalues of AA^T and $A^TA)$: D contains their square roots along the $(r \times r)$ initial sub-diagonal.

$$\text{And } U = \left[\begin{array}{ccc|ccc} \\ & \underbrace{u_1 \cdots u_r}_{} & & & \underbrace{u_{r+1} \cdots u_m}_{} & \\ \\ \end{array}\right]$$

This is $m \times m$

First r columns are eigenvectors of AA^T corresponding to nonzero eigenvalues $\lambda_1^2, \dots, \lambda_r^2$.

THIS IS A BASIS FOR THE COLUMN SPACE $C(A)$

Columns $r+1$ through m are eigenvectors corresponding to the $m-(r+1)$ copies of the zero eigenvalue of AA^T

THIS IS A BASIS FOR THE LEFT NULLSPACE $N(A^T)$

And similarly,

$$V = \left[\begin{array}{ccc|ccc} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_m \end{array}\right]$$

First r columns are eigenvectors of A^TA corresponding to $\lambda_1^2, \ldots, \lambda_r^2$

THIS IS A BASIS FOR THE ROW SPACE $C(A^T)$

Next $n-(r+1)$ columns are eigenvectors of AA^T corresponding to the zero eigenvalues.

THIS IS A BASIS FOR THE NULL SPACE $N(A)$

Remember: V is TRANSPOSED in $A = UDV^T$

PROOF OF SVD THEOREM

First, let's see what we want to prove:

$$A = UDV^T$$

Since V is orthogonal [by spectral theorem!], $V^T = V^{-1}$, so it is enough to show

$$AV = UD$$

Now, $AV = A \left[v_1 \cdots v_r \mid v_{r+1} \cdots v_n \right]$

Let's do this in two stages:

- If i is in $\{1, \ldots, r\}$ then

$$A^T A v_i = \lambda_i^2 v_i \quad \text{with } \lambda_i \neq 0.$$

So, $A A^T A v_i = A \lambda_i^2 v_i$

So, $A A^T (A v_i) = \lambda_i^2 (A v_i)$

Meaning: Av_i is an eigenvector of AA^T with eigenvalue λ_i^2: so, $Av_i = c u_i$ for some scalar $c \neq 0$.

But $c^2 = \|Av_i\|^2 = v_i^T A^T A v_i = \lambda_i^2$,

So $c = \lambda_i$

Meaning, $\boxed{Av_i = \lambda_i u_i}$ as desired.

- If i is in $\{r+1, \ldots, n\}$
  then both sides are zero!